Generalized almost disjoint families and injective Banach spaces

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Joint work in progress with David Schrittesser, begun during a trip to Harbin, China during October 2023.

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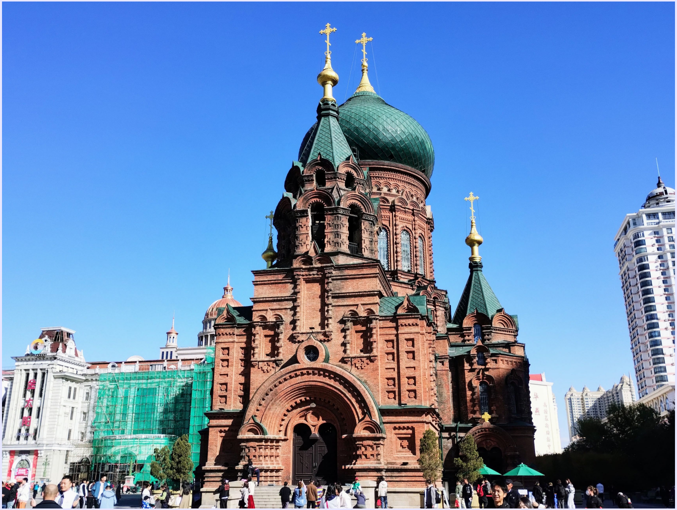

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A Banach space E is injective iff it is a complemented subspace of $\ell^\infty(\Gamma)$ for some set Γ, i.e., $\ell^\infty(\Gamma) \cong E \bigoplus X$ for some space X.

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$$\sigma\left(\sum_{A \in \mathcal{B}} [1_A] \right)(m) > 0.99 |\mathcal{B}|.$$

Thus, σ takes elements of the unit ball of ℓ^∞ / c_0 to elements of ℓ^∞ of arbitrarily high norm, contradicting the fact that σ is continuous (and hence bounded). $\square$

One more step

We can do better, using the following theorem of Rosenthal.

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If E is an injective Banach space, Γ is a set, and E contains a subspace isomorphic to $c_0(\Gamma)$, then it contains a subspace isomorphic to $\ell^\infty(\Gamma)$.

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In the notation of the previous proof, $\{[1_A] \mid A \in \mathcal{A}\}$ generates a copy of $c_0(\mathcal{A})$ in ℓ^∞/c_0. We can take $|\mathcal{A}| = 2^{\aleph_0}$. If ℓ^∞/c_0 were injective, it would then contain a copy of $\ell^\infty(2^{\aleph_0})$, but it is too small for this. $\square$

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If $K = \omega + 1$, then a generalized almost disjoint family in K is simply a classical (nontrivial) almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$.

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If D is a dense subset of K, then $C(K)$ embeds as a closed subspace of $\ell^\infty(D)$. This leads us to be interested in quotient spaces of the form

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    1. $b \subseteq^* a_{x_0} \cup \ldots \cup a_{x_n}$ for some $x_0, \ldots, x_n \in {}^{<\omega_1}2$; or
    2. for all $x_0, \ldots, x_n \in {}^{<\omega_1}2$, there is $y \in {}^{<\omega_1}2$ incompatible with each x_i such that $|a_y \cap (b \setminus (a_{x_0} \cup \ldots \cup a_{x_n})| = \aleph_0$.

Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\alpha} \mid \alpha < \omega_1 \rangle$ of elements of $[\omega]^\omega$.

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Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\alpha} \mid \alpha < \omega_1 \rangle$ of elements of $[\omega]^\omega$. Let A_f be the collection of all $\mathcal{U} \in \mathbb{N}^*$ for which there exists $\alpha < \omega_1$ such that $a_{f\restriction\alpha} \in \mathcal{U}$.

Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\alpha} \mid \alpha < \omega_1 \rangle$ of elements of $[\omega]^\omega$. Let A_f be the collection of all $\mathcal{U} \in \mathbb{N}^*$ for which there exists $\alpha < \omega_1$ such that $a_{f\restriction\alpha} \in \mathcal{U}$. Then $\{A_f \mid f \in {}^{\omega_1}2\}$ is a generalized almost disjoint family. $\square$

Back to c_0

Fact

ℓ^∞ / c_0 is isomorphic to $C(\mathbb{N}^*)$.

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$$0 \to c_0 \xrightarrow{\iota_0} \ell^\infty \xrightarrow{\iota_1} \ell^\infty(2^{\aleph_0}) \to \cdots$$

such that

$$\ell^\infty(2^{\aleph_0})/\mathrm{im}(\iota_1) \cong \ell^\infty(D)/C(\mathbb{N}^*).$$

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Thus, if $\ell^\infty(D)/C(\mathbb{N}^*)$ is not injective, then the injective dimension of c_0 is at least 3.

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If CH holds, then $\mathbb{N}^*$ contains a generalized AD family of size $2^{\aleph_1}$. Thus, $\ell^\infty(D)/C(\mathbb{N}^*)$ contains a copy of $c_0(2^{\aleph_1})$.

Theorem

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Proof.

If CH holds, then $\mathbb{N}^*$ contains a generalized AD family of size $2^{\aleph_1}$. Thus, $\ell^\infty(D)/C(\mathbb{N}^*)$ contains a copy of $c_0(2^{\aleph_1})$. If $\ell^\infty(D)/C(\mathbb{N}^*)$ were injective, then it would contain a copy of $\ell^\infty(2^{\aleph_1})$, but it is too small for this, since

$$|\ell^\infty(2^{\aleph_1})| = 2^{2^{\aleph_1}}$$

but

$$|\ell^\infty(D)/C(\mathbb{N}^*)| = 2^{2^{\aleph_0}}.$$

□

Questions

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Can the cardinal arithmetic assumptions be removed from these results?

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Is the injective dimension of c_0 infinite?

Thank you!